Solve Initial Value ODE Using 4^{th} Order Runge-Kutta (RK) Method

Objectives

- Solve a First order ODE Using 4th Order Runge – Kutta (RK) Method
- Runge-Kutta methods are a family of single-point methods
- Runge-Kutta methods are explicit methods
- The Fourth – Order Runge – Kutta (RK) Method is a popular 4^{th} order method.

The Fourth – Order Runge – Kutta Method

- Consider a general nonlinear first order ODE of the form
- $y' = f(t, y)$, $y(t_0) = y_0$
- $y_{n+1} = y_n + \left(\frac{1}{6}\right) * (\Delta y_1 + 2 * \Delta y_2 + 2 * \Delta y_3 + \Delta y_4)$
- $\Delta y_1 = h * f(t_n, y_n)$
- $\Delta y_2 = h * f\left(t_n + \frac{h}{2}, y_n + \frac{\Delta y_1}{2}\right)$
- $\Delta y_3 = h * f\left(t_n + \frac{h}{2}, y_n + \frac{\Delta y_2}{2}\right)$
- $\Delta y_4 = h * f(t_n + h, y_n + \Delta y_3)$

The Fourth – Order Runge – Kutta Method

- The (Finite Difference Equations) FDEs are explicit and require four derivative function evaluations per step.
- The FDEs are consistent, $O(\Delta t^5)$ locally and $O(\Delta t^4)$ globally.
- The FDEs are conditionally stable (i.e., $\alpha \, \Delta t <= 2.785...$).
- The FDEs are consistent and conditionally stable, and thus, convergent.
- Algorithms based on the repetitive application of Runge-Kutta FDEs are called Runge-Kutta methods.

The Fourth – Order Runge – Kutta Method

- Example ODE Problem
- $\frac{dy}{dx}$ = - 2 * x^3 + 12 * x^2 - 20 * x + 8.5
- From x = 0 to 4 with a step size of 0.5;
- The initial condition at *x* = 0 is *y* = 1.
- The exact solution is given as
- y = - 0.5 * x^4 + 4 * x^3 - 10 *x^2 + 8.5 * x + 1

The Fourth – Order Runge – Kutta Method

- f_n = - 2 * x^3 + 12 * x^2 - 20 * x + 8.5 ; y(x=0) = 1;
- Let n = 0;
- $\Delta y_1 = h \ * f(t_n, y_n) = h \ * f(t_0, y_0) = h \ * f(0,1)$

  = 0.5 * (- 2 * 0^3 + 12 * 0^2 - 20 * 0 + 8.5) = 4.25

- $\Delta y_2 \ = h \ * f\left(t_n + \frac{h}{2}, y_n + \frac{\Delta y_1}{2}\right) = h \ * f\left(t_0 + \frac{0.5}{2}, y_0 + \frac{\Delta y_1}{2}\right)$

  $= h \ * f\left(0.25, y_0 + \frac{\Delta y_1}{2}\right)$

  = 0.5 * (- 2 * 0.25^3 + 12 * 0.25^2 - 20 * 0.25 + 8.5)

  = 0.5 * (- 0.03125 + 0.75 – 5 + 8.5) = 2.109375

The Fourth – Order Runge – Kutta Method

- $\Delta y_3 = h * f\left(t_n + \frac{h}{2}, y_n + \frac{\Delta y_2}{2}\right) = h * f\left(t_0 + \frac{0.5}{2}, y_0 + \frac{\Delta y_2}{2}\right)$

$= h * f\left(0.25, y_0 + \frac{\Delta y_2}{2}\right)$ = 2.109375 (same as previous step)

- $\Delta y_4 = h * f(t_n + h, y_n + \Delta y_3) = h * f(t_0 + 0.5, y_0 + \Delta y_3)$

$= h * f(0.5, y_0 + \Delta y_3)$ = 0.5 * (- 2 * 0.5^3 + 12 * 0.5^2 - 20 * 0.5 + 8.5)

= 0.5 * (- 0.25 + 3 – 10 + 8.5) = 0.625

The Fourth – Order Runge – Kutta Method

- $y_{n+1} = y_n + \left(\frac{1}{6}\right) * (\Delta y_1 + 2 * \Delta y_2 + 2 * \Delta y_3 + \Delta y_4)$
- Let n = 0
- $y_1 = y_0 + \left(\frac{1}{6}\right) * (\Delta y_1 + 2 * \Delta y_2 + 2 * \Delta y_3 + \Delta y_4)$
- $y_1 = 1 + \left(\frac{1}{6}\right) * (4.25 + 2 * 2.109375 + 2 * 2.109375 + 0.625)$
- y_1 = 3.21875
- Likewise, y_2, y_3 etc can be evaluated

Summary

In this video,

- We presented the Fourth Order Runge-Kutta method to solve an Initial Value ODE
- The $\text{Fourth Order Runge} - \text{Kutta method}$ is conditionally stable.
- The global error is $O(\Delta t^4)$.
- In the next video, we can look at a system of ODEs